$$R = \frac{wh^2}{2} \sin c.$$

When $\theta = 90°$, the general equations become

$$\tan \phi = \frac{\tan \beta_1 \sqrt{(\tan \delta_1)}}{\sqrt{(\tan \delta_1)} - \sqrt{\{(\tan \beta_1 + \tan \delta_{11})\}}}, \qquad (20)$$

$$R = \frac{wh^2 \sin \beta_1 \tan \beta_1}{2 \cos \delta_1} \left(\frac{1}{\sqrt{(\tan \beta_1 + \tan \delta_1)} - \sqrt{(\tan \delta_1)}} \right)^2. \qquad (21)$$

If the force in this case be supposed to act horizontally ($\delta_1 + \beta_1 = 90°$), these equations may be reduced to

$$\tan \phi = \cot \left(c - \frac{\beta}{2} \right); \qquad (22)$$

$$R = \frac{wh^2}{2} \cot^2 \left(c - \frac{\beta}{2} \right). \qquad (23)$$

If the face be vertical, then $\beta = c$, and the equations may be further reduced to

$$\tan \phi = \cot \tfrac{1}{2} c; \qquad (24)$$

$$R = \frac{wh^3}{2} \cot^2 \tfrac{1}{2} c. \qquad (25)$$

The Rev. Charles Graves communicated the following note respecting geodetic lines on surfaces of the second order.

"At a meeting of the Academy which took place in last June, I stated a general theorem, from which I am able to deduce Joachimsthal's theorem respecting the geodetic lines traced on a central surface of the second order; and at the same time to show geometrically the reason why the property enunciated in it is common to geodetic lines and to lines of curvature. From the general theorem to which I refer, the following proposition is a corollary:

"*If a central surface of the second order* (A) *be circumscribed by a cone* (a), *the quantity* PD *is the same for* L, L′, L″, L‴, *four sides of the cone which make equal angles with its internal axis:* P *denoting the perpendicular from the centre*

of the surface on the tangent plane passing through one of those sides, and D *the semidiameter parallel to that side.*

"If V, the vertex of the cone, be supposed to approach indefinitely near to a point on the surface, its axes ultimately coincide with the normal at that point, and the tangents to the lines of greatest and least curvature passing through it. The plane of two successive elements of a geodetic line through V contains the normal, and those elements are equally inclined to it. So likewise the plane of two successive elements of a line of curvature passing through V contains the tangent to that line of curvature at V; and the elements themselves are equally inclined to the normal. In virtue of the preceding proposition we are, therefore, entitled to conclude, that the quantity PD remains unaltered for two successive elements, either of a geodetic line or of a line of curvature traced on a central surface of the second order.

"Returning to the case in which the vertex of the cone is at a finite distance from the surface, we may now say that

"*If a central surface of the second order* (A) *be circumscribed by a cone* (a), *the quantity* PD *is the same for the geodetic lines which are the prolongations of* L, L′, L″, L‴, *four sides of the cone which make equal angles with its internal axis.*

"In what follows I shall suppose the surface (A) to be an ellipsoid, in order to avoid the enumeration of a variety of cases. The conclusions arrived at may, however, be adapted to the hyperboloids by obvious modifications.

"The four sides of the cone, denoted according to their order by L, L', L'', L''', being all tangents to geodetics on the ellipsoid for which PD is the same,* are likewise all tangents to the same confocal hyperboloid (B), which intersects the surface (A) in the pair of opposite lines of curvature

* I think it would be convenient, in future, to designate geodetic lines of this kind as *similar* geodetics.

touched by the geodetics. The cone (b), therefore, which envelopes (B), and has the same vertex as (a), is confocal with (a), and intersects it orthogonally. The normal planes to (a) along L and L'', being thus tangent planes to (b), intersect in a right line drawn from V to the pole of the plane LL'' with relation to (B). Moreover, this right line lies in a plane perpendicular to the internal axis of the cone (a), and therefore makes equal angles with L and L''.

" Suppose now that the straight lines L and L'' be replaced by a continuous flexible and inextensible cord, which is kept stretched by a style at V, and prolonged in the direction of geodetic lines to two fixed points p, p'', at which it is attached to the surface: it is easy to show that the style will trace a curve on an ellipsoid (A') passing through V, and confocal with (A); whilst it moves in such a manner as to allow the cord to roll on one, and off the other, geodetic line. In fact the path described by the style at the beginning of its motion, if any motion be possible under the prescribed conditions, must be in the intersection of the two planes through L and L'', which are normal to (a): and we have already seen that this intersection is in a plane perpendicular to the internal axis of the cone (a), that is, in the tangent plane to a confocal ellipsoid passing through V. But further, motion is possible, though the length of the cord remains unaltered; since the two straight parts of it are equally inclined to the line of intersection of the two normal planes. From what has been said above we may derive a simple mode of determining the direction of the tangent to the curve traced on (A') at the point V. For this purpose we must draw a right line from V through the pole of the plane of the two straight portions of the cord, taken with relation to the hyperboloid (B).

" What has been already proved with respect to L and L'' holds good in like manner for L' and L'''. And it is to be observed that the paths described by the style on (A'), corresponding respectively to these two pairs of opposite sides,

though not the same, are equally inclined to the lines of curvature on (A') passing through V. This suggests the theorem, that if the plane of L and L'' be a principal plane of the cone, the path of the style will touch a line of curvature on (A') at V.

"Let us next consider a pair of adjacent sides of the cone, such as L and L'. Normal planes to the cone (a) along these sides intersect in a right line, which lies in a plane perpendicular to that external axis of the cone (a), through which the plane of L and L' passes. Hence it follows, as before, that two intersecting cords, L and L', may be both rolled on, or both rolled off the geodetic lines upon which we suppose them prolonged to fixed points, p and p', in such a manner that the shortest distances between their intersection and the fixed points p, p', shall have a constant *difference*. And their intersection at V will lie upon a hyperboloid (B'), confocal with the ellipsoids (A) and (A').

"Lastly, considering the pair of sides L and L''', and cords produced along them to fixed points p, p''', on the geodetic lines touched by them, we see that if the difference between the lengths Vp, Vp''' remain constant, V will trace a curve on a second hyperboloid (C'), which passes through V, and is confocal with (A') and (B').

"It is obvious that the curve described by the point V, under the circumstances considered above, is not, in general, a geodetic line on a surface confocal to (A). We may, however, regulate the motion of the cords so as to effect this.

"For instance, let the four cords, L, L', L'', L''', be prolonged in the direction of similar geodetics until they touch two opposite lines of curvature, along which they are thenceforth applied and carried on to fixed points, p, p', p'', p'''. Then a style at V, stretching a continuous cord pVp'', which coincides with two *opposite* sides of the cone, L and L'', will trace *a geodetic line* upon the confocal ellipsoid (A'), provided it be made to move always in the plane of the two straight portions

of the cord; whilst the parts which coincide with geodetic lines on (A) roll, one of them on, and the other off, its corresponding line of curvature.

"If, on the other hand, we consider a continuous cord $p\,Vp'$ coincident with a pair of *adjacent* sides of the cone, as L and L', we see that the locus of the style at V, which keeps it stretched, will be *a line of curvature* on a confocal ellipsoid, if the conditions of motion be the same as before.

"The theorems here announced are meant to take the place of two which were incorrectly given at page 192. I fell into an error in the statement of them, partly by my haste in generalizing from particular cases in which they are true; and partly in consequence of my having formed an inaccurate conception of the *form* of geodetic lines in general.

"For clearer views as regards this latter point, I acknowledge myself indebted to the recently published researches of my friend, Dr. Hart.

"I may be allowed to take the present opportunity of mentioning, that a theorem lately announced by him to the Academy, respecting the form of a geodetic line which passes through an umbilic, may be derived geometrically from a theorem discovered by Mr. Michael Roberts.

"Mr. Roberts has shown that if two geodetic lines be drawn from the interior umbilics of a line of curvature of an ellipsoid to the same point on the curve, the product of the tangents of the halves of the angles which they make with the arc of the principal ellipse joining the umbilics is constant.

"Suppose now that the line of curvature referred to inth e preceding proposition is a principal section passing through the extremities of the mean axis. Let U and U_1 be its interior umbilics, and U', U_1' the umbilics diametrically opposite. Geodetic lines drawn from U and U_1 to a point S, taken anywhere in this principal section, make with the arc UU_1 angles SUU_1, SU_1U, the product of the tangents of whose halves is constant. Prolong either of these geodetics

US to the opposite umbilic U'. Then, by reason of the symmetry of the surface, we shall have the angle $SU'U$ equal to SU_1U', and supplemental to SU_1U. Consequently the tangents of the halves of SUU_1 and $SU'U_1$ are to each other in a constant ratio. From this it appears that the principal ellipse passing through the umbilics is a common asymptot to all the geodetic lines which pass through either umbilic.

"Though the umbilical geodetics are thus shown to be infinite spires, it is not true that all the geodetics on the surface of an ellipsoid are of the same nature. Besides the geodetics which coincide with the principal sections of the surface, there are others among them which are closed curves. An example will make this evident.

"A circular disc, with a regular figure of an even number of sides inscribed in it, may be regarded as an infinitely flattened spheroid of revolution, with a continuous geodetic line traced upon its surface. In fact a closed cord, carried along in the direction of the sides of the regular figure, and passing over at each angle to the opposite side of the disc, would be kept stretched round it.

"I hope to be able, before long, to communicate to the Academy a series of remarkable results respecting the comparison of similar geodetic arcs, at which I have arrived by means of the theorems stated in the beginning of the present note. I expect also, by the translation of these geometrical theorems into analytical language, to obtain some new relations between the integrals, to the consideration of which we are led in the rectification of the geodetic lines and lines of curvature of a surface of the second order."

Rev. William Roberts communicated an analytic proof of the theorem stated by Mr. Graves, and made some observations on different applications of the formula of M. Liouville, on which the proof depends.